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Keywords

fractional integrals, fractional calculus, intermediate electromagnetic waves

Comments

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Phase and Amplitude of Fractional-Order Intermediate Wave[†]

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Key Words: Fractional Integrals, Fractional Calculus, Intermediate Electromagnetic Waves.

Abstract

The behavior of the amplitude and phase of the "intermediate wave", which we previously introduced as certain fractional solutions to the standard scalar Helmholtz equation [1], is addressed and presented. These waves effectively behave as intermediate cases between the canonical cases of the plane-wave and cylindrical wave propagation. We show that the amplitude and phase of such intermediate wave undergo interesting "evolutions" as the fractionalization parameter ν attains fractional values between zero and unity. Possible extension into the novel concept of intermediate guided-wave geometries is just speculated.

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Introduction

Recently, we considered the possibility of using the method of fractional integration/differentiation in finding certain "fractional" or "intermediate" solutions for the standard scalar Helmholtz equation [1]. It is well known that for the scalar Helmholtz equation the canonical solutions for the one-, two- and three-dimensional cases are identified as plane, cylindrical, and spherical waves, respectively, with the corresponding sources as one-, two- and three-dimensional Dirac delta functions. We then asked the following questions: Can there be a solution for the scalar Helmholtz equation that effectively behaves as the "intermediate" case between these canonical cases, for example, an "intermediate" wave between the cases of a plane wave and a cylindrical wave? What kind of source distributions would effectively behave as such intermediate steps between the integer-dimensional Dirac delta functions, e.g., between the one- and two-dimensional Dirac delta functions? In our previous work, we showed that fractional integration/differentiation, which are mathematical tools studied in the field of fractional calculus (see, e.g. [2], [3]), can be utilized to find the "intermediate" sources and waves that satisfy the conventional scalar Helmholtz equation [1]. These solutions, which depend on the fractionalization parameter ν , have properties that are *effectively* intermediate between the cases represented by the integer-order ν . In the present work, we show the behavior of phase and amplitude of such intermediate waves in the far-zone region, as the parameter ν attains fractional values between zero and unity where $\nu=0$ represents the case of the cylindrical wave propagation and $\nu=1$ denotes the plane wave propagation. As will be seen, the phase and amplitude effectively "evolve" between the two canonical cases of plane and cylindrical waves. The present work is one of the results of our general efforts in recent years to explore potential utilities and possible physical implications of the mathematical machinery of fractional derivatives and fractional integrals in electromagnetism ([1], [4]-[8]). We have applied the tools of fractional derivatives/integrals in several specific electromagnetic case

studies, and have obtained promising results that demonstrate that these mathematical operators may be interesting and useful tools in electromagnetic theory.

"Intermediate Wave" between a plane wave and a cylindrical wave

In our previous work, we derived a solution to the scalar Helmholtz equation that we named the "intermediate" or "fractional" solution. Here we provide a brief review of that solution. The details can be found in [1]. To describe the geometry of the problem, let us consider a Cartesian coordinate system (x, y, z) in a three-dimensional physical space. The Green's function for the scalar Helmholtz equation satisfies the following equation

$$\nabla^2 G(\vec{r}) + k^2 G(\vec{r}) = -\delta(\vec{r}) \quad (1)$$

where $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is the position vector for the observation point, \hat{a}_x , \hat{a}_y , and \hat{a}_z are the unit vectors in the coordinate system, and k is a scalar constant for a homogeneous isotropic space. It is well known that the solution to Eq. (1) for a one-dimensional Dirac delta function source $\delta_1(\vec{r}) \equiv \delta(x)$ located at the y - z plane is a plane wave [9, p. 811] given as

$$G_1(\vec{r}) = G_1(x) = i \frac{e^{ik|x|}}{2k}. \quad (2)$$

When the source is a two-dimensional Dirac delta function $\delta_2(\vec{r}) \equiv \delta(x)\delta(y)$ (i.e., a uniform line source along the z -axis), the solution is a cylindrical wave [9, p. 811] as

$$G_2(\vec{r}) = G_2(x, y) = \frac{i}{4} H_o^{(1)}(k\sqrt{x^2 + y^2}) \quad (3)$$

where $H_o^{(1)}(\cdot)$ is the zeroth-order Hankel function of the first kind [10, ch.9]. Through certain mathematical steps in our previous work [1], we have shown that a source that is described as:

$$_e S_\nu(x, y) = (1/2) \left[{}_{-\infty} D_y^{-\nu} \delta(x)\delta(y) + {}_{-\infty} D_{-y}^{-\nu} \delta(x)\delta(y) \right] = \frac{\delta(x)|y|^{\nu-1}}{2\Gamma(\nu)} \quad (4)$$

for $0 < \nu < 1$

can be considered as an "intermediate" source between the two cases of one- and two-dimensional Dirac delta function sources [1].¹ See Fig. 1. The subscript ν is a parameter with fractional values between zero and unity, the pre-subscript "e" indicates the "even" (i.e., symmetric) nature of this source with respect to the y -coordinate, and the symbol ${}_{-\infty}D_y^{-\nu}$ represents Riemann-Liouville Fractional Integration defined in references on fractional calculus (e.g., [2], p. 49) as

$${}_aD_u^\beta f(u) \equiv \frac{1}{\Gamma(-\beta)} \int_a^u (u-w)^{-\beta-1} f(w) dw \quad \text{for } \beta < 0 \quad \text{and} \quad u > a \quad (5)$$

which gives the fractional β -order integral of function $f(u)$ with the lower limit of integration being a , and $\Gamma(\cdot)$ is the Gamma function. The source in Eq. (4), which involves fractional integrals (of the order $-\nu$) of the two-dimensional Dirac delta function $\delta(x)\delta(y)$, is a distributed source in the y - z plane. (In the fractional integration given in Eq. (4), the lower limit of integration in the definition of the Riemann-Liouville integral is taken to be $a = -\infty$.) When $\nu \rightarrow 0$, the above source approaches the two-dimensional Dirac delta function located at $x = y = 0$ (i.e., along the z -axis), and when $\nu \rightarrow 1$ the source in Eq. (4) becomes $\frac{1}{2}\delta(x)$, which is a one-dimensional source located at $x = 0$ (i.e., in the y - z plane). With the above "intermediate" source as the source term in the right-hand side of Eq. (1), the solution to this equation for the observation points on the symmetry plane (i.e. on the x - z plane where $y = 0$) and for $x > 0$ can be exactly obtained and written [1] as¹

$$\Psi_\nu(x, y = 0) = \frac{i \Gamma\left[\frac{1-\nu}{2}\right] \cos\left(\frac{\nu\pi}{2}\right)}{4\sqrt{\pi}} \left(\frac{x}{2k}\right)^{\nu/2} H_{-\nu/2}^{(1)}(kx) \quad (6)$$

for $x > 0$ and for $0 < \nu < 1$

¹ As in [1], here again for the sake of mathematical simplicity we assume that all quantities such as x , y , and z coordinates, k , and Ψ_ν are physically dimensionless. Thus, the fractional integration operator in Eq. (4) would not cause any inconsistency in dimensions of the quantities. However, if these quantities had to possess physical dimensions, one would then need to multiply the fractional integral operator ${}_{-\infty}D_y^{-\nu}$ in Eq. (4) by an unimportant constant $l^{-\nu}$ with l having a dimension of the y coordinate (e.g., length). Such a multiplicative constant would then also appear in Eq. (6).

where $H_{-\nu/2}^{(1)}$ is the Hankel function of order $-\nu/2$ and of the first kind [10, p. 358]. For observation points outside the x - z plane, we have found the approximate solution suitable for the region where $k\rho \gg 1$ and $\varphi \neq 0$ (but φ not too small), with ρ and φ being shorthands for $\sqrt{x^2 + y^2}$ and $\tan^{-1}(y/x)$, respectively [1]. This approximate solution is given as¹

$${}_e\Psi_\nu(x, y) \underset{k\rho \gg 1}{\cong} \frac{i}{4\pi} \cos\left(\frac{\nu\pi}{2}\right) (k \sin|\varphi|)^{-\nu} \sqrt{\frac{2\pi}{k\rho}} e^{ik\rho - i\pi/4} + \frac{i}{4k^\nu \Gamma(\nu)} \frac{e^{ik|x|}}{(k|y|)^{1-\nu}}. \quad (7)$$

for $0 < \nu < 1$,

$k\rho \gg 1$ and $\varphi \neq 0$ but φ not too small

Equations (6) and (7) describe the "intermediate wave" between the cases of the plane and cylindrical wave. As evident in Eq. (7), this expression has two parts: a cylindrical wave whose magnitude drops as $\rho^{-1/2}$ in the far zone, and an inhomogeneous plane wave which propagates in the x direction but its amplitude drops with y as $|y|^{\nu-1}$ for $0 < \nu < 1$. When $\nu \rightarrow 0$, the source given in Eq. (4) becomes a two-dimensional Dirac delta function at $x = y = 0$, the second term in Eq. (7) representing a "plane wave" portion of the solution disappears owing to $\Gamma(\nu)$ in its denominator, and the first term becomes $\frac{i}{4} \sqrt{\frac{2}{\pi k\rho}} e^{ik\rho - i\pi/4}$ which is the asymptotic form for $\frac{i}{4} H_o^{(1)}(k\rho)$, as expected. For $\nu \rightarrow 1$, the source in Eq. (4) becomes $(1/2)\delta(x)$, the cylindrical term in Eq. (7) disappears, and the plane-wave part of Eq. (7) becomes $(i/4k) \exp(ik|x|)$ which is half of the value of $G_1(x)$ when the source is $\delta(x)$. So the above solution effectively behaves as an "intermediate wave" between the cases of plane and cylindrical waves. As we have indicated in [1], for the far-field observation points along the x - z plane, the magnitude of the exact solution given in Eq. (6) can be given as

$$|\Psi_\nu(x, y=0)| \underset{kx \gg 1}{=} \frac{\Gamma\left[\frac{1-\nu}{2}\right] |\cos(\nu\pi/2)|}{4\pi 2^{(\nu-1)/2} k^{(\nu+1)/2}} |x|^{(\nu-1)/2} \quad \text{for } 0 < \nu < 1, \text{ and for } kx \gg 1 \quad (8)$$

which illustrates that the magnitude of the solution on the x - z plane drops as $|x|^{(\nu-1)/2}$ in the far zone. So *effectively*, the two- and one-dimensional Green's functions of Eq. (1) have been "smoothly connected" by varying the order of fractional integration of the two-dimensional delta functions in the source term Eq. (4).

Here we should point out that the "intermediate" source given in Eq. (4) is not simply a collection of *one* single line source (for cylindrical wave) and *one* single sheet source (for a plane wave) with appropriate coefficients. Because if we had such a combination, the far-zone field would have been dominated by the plane wave. However, for our source the wave in the far zone behaves as an *intermediate* case between the *plane* and *cylindrical* waves.

Phase and Amplitude of Intermediate Wave

In order to gain physical insights into radiation properties of a source, one ordinarily studies the radiation pattern, i.e., the angular variation of the far-zone radiation field (or radiated energy) of the source. However, in the present case the field expression given in Eq. (7) does not possess a common radial dependence since it has both the cylindrical-wave and the plane-wave portions. Therefore, instead we present graphically the behavior of the phase and amplitude of this field expression. Treating Eq. (7) as a complex quantity with the amplitude A_ν and the phase P_ν both being dependent on the coordinates of the observation point and on the parameters k and ν , we can recast Eq. (7) as

$${}_e\Psi_\nu(x, y) = A_\nu(x, y) \exp[i P_\nu(x, y)]. \quad (9)$$

$kp \gg 1$

Using MATLAB[®] (MathWorks, Inc.) we plot the Phase P_ν and the amplitude A_ν of Eq. (7) in a given region in the x - y plane. Figures 2 and 3 present such plots for the phase P_ν

and the amplitude A_ν , respectively.² These plots reveal several interesting features of the far-zone field that are described below:

As can be seen from Fig. 2, which shows the loci of constant phase in a square region in the x - y plane ($3\pi \leq kx \leq 21\pi$ and $3\pi \leq ky \leq 21\pi$)³, when $\nu \rightarrow 0$ we get a cylindrical wave and the loci of the constant phase in this square region are therefore portions of concentric circles. For this case, the amplitude plot given in Fig. 3 shows a variation proportional to $\rho^{-1/2}$, as expected. As ν increases, one notices some deformation in the loci of the constant phase in this region. This is due to the contribution of the plane-wave portion of Eq. (7) to originally dominating cylindrical-wave portion. A corresponding deformation is also noticed in the amplitude plot in Fig. 3. As ν increases even further, the contribution of the cylindrical portion generally decreases and the plane-wave part contributes more. In this process, one notices certain locations in the phase plots where the loci of constant phases get together and pass through these common locations. For example, at $\nu = 0.45$, we see from Fig. 2 that there are places in the plot where loci of differing values of phases meet together. By further increasing the parameter ν , the phase loci, which originally were part of the concentric circles and then have gone through the "breakage" locations, start to get connected to the neighboring loci in order to form new loci that are becoming more and more like a straight line as ν approaches unity. Finally when $\nu = 1$, the phase loci become straight lines parallel with the y -axis indicating plane wave propagation in the x direction. Figure 3 presents the plots of amplitudes corresponding to the cases given in Fig. 2. This

² In these plots, kx and ky are obviously dimensionless, and for the sake of simplicity we take k to be unity. This latter assumption does not affect the behavior of the phase plots and the general shape of the amplitude plots, because in Eq. (7) aside from the kx , ky , and $k\rho$ terms, the terms involving k appear as multiplicative constant $k^{-\nu}$. See also footnote 1.

³ For the region in the x - y plane over which we present the phase and amplitude of intermediate wave given in Eq. (7), one should consider the two conditions $k\rho \gg 1$ and $\varphi \neq 0$ but not too small – the conditions under which Eq. (7) was obtained [1]. For the specific square region selected as an example given here, the smallest $k\rho$ turns out to be about $3\pi\sqrt{2} \approx 13.33$. The smallest angle φ for this region is $\varphi = \tan^{-1}(1/7) \approx 8.13^\circ$.

"evolution" of phase and amplitude is indeed necessary in order to "connect" the two limiting cases of the cylindrical and the plane wave propagation. These plots provide us with insights into the behavior of the "intermediate wave".

One of the interesting issues to pursue is whether the concept of "intermediate wave" can be extended to "intermediate guided-waves structures". In other words, it would be interesting to explore the possibility of having intermediate waveguide geometries whose guidance characteristics straddles its two limiting cases. In other words, what should be the shape of a waveguide that is the "intermediate shape" between a parallel-plate waveguide and a circular-cylindrical waveguide? Since in a parallel-plate waveguide the modes can be constructed by using superposition of *plane* waves bouncing back and forth between the two walls of this waveguide, and for the circular cylindrical waveguides modes with azimuthal symmetry can be constructed by superposition of *cylindrical* waves whose phase fronts have conical shapes with the waveguide axis as their axis of symmetry, it would be interesting to inquire whether one can have a waveguide geometry whose modes can be described as superposition of the intermediate waves we have introduced. Returning to the parallel-plate and circular-cylindrical waveguides, it is clear that in these waveguides, the walls are consistent with the nature of the constituent waves that can form the guided modes in the respective waveguides, i.e., in parallel-plate waveguide the walls are planar (and the constituent waves are also plane waves), and in the circular-cylindrical waveguides, the walls are circular cylindrical (and the constituent waves are cylindrical). So, one may speculate that if we can construct a waveguide whose wall's shape is consistent with the shape of the phase loci of the intermediate wave, may such a waveguide effectively behave as an "intermediate" guided structure whose characteristics is between the two waveguides? These are among the topics currently under study by the author.

Acknowledgements

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Figure Captions

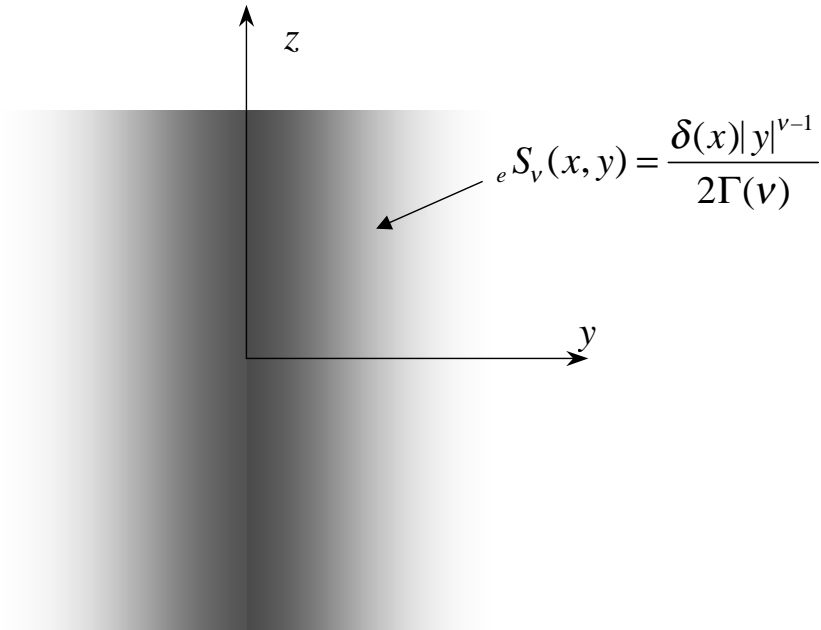
Fig. 1. A sketch of the source ${}_eS_\nu(x, y)$. This is a sheet source located in the plane $x = 0$. It is independent of z coordinate, but varies with y according to Eq. (4). For $\nu \rightarrow 0$ this source becomes a 2-D Dirac delta function $\delta(x)\delta(y)$, while for $\nu \rightarrow 1$ it becomes $\frac{1}{2}\delta(x)$. (All the quantities are assumed to be physically dimensionless. See footnote 1.)

Fig. 2. Contour plots of the phase of the fractional-order “intermediate” wave of Eq. (7) for $0 \leq \nu \leq 1$. In these plots, loci of constant phase are shown in the square region $3\pi \leq kx \leq 21\pi$ and $3\pi \leq ky \leq 21\pi$ in the x - y plane. Here and in Fig. 3, k is taken to be unity (See footnote 2). For the two limiting cases of $\nu = 0$ and $\nu = 1$, the phase loci represent the cases of cylindrical and plane wave, respectively, as expected. As the parameter ν takes the values between zero and unity, the evolution of intermediate wave is clearly observed in the above plots. For the square region used here, when the fractional parameter ν attains values around 0.45 the evolution of the phase becomes particularly interesting due to the “breakage” of the loci of constant phase, as seen above. Therefore, in addition to the two limiting values of zero and unity, the contour plots are shown for values of ν around 0.45, i.e., $\nu = 0.4, 0.45, 0.5$, and 0.6 . The “breakage” of the phase loci may happen at a different value of ν if one looks at the phase at a different region in the x - y plane.

Fig. 3. Plots of the amplitude of the fractional-order “intermediate” wave of Eq. (7) for $0 \leq \nu \leq 1$. As in Fig. 2, these plots are given for the square region $3\pi \leq kx \leq 21\pi$ and $3\pi \leq ky \leq 21\pi$ in the x - y plane. Note that the vertical scales are not the same in all the

plots, and also note that in this view of the mesh plots the point $(kx = 3\pi, ky = 3\pi)$ is in the back. The values of parameter ν are chosen to be the same used in Fig. 2. For the two limiting cases of $\nu=0$ and $\nu=1$, the amplitude plots clearly show the cases of cylindrical and plane wave, respectively. Like in Fig. 2, for values of ν between zero and unity, one can see the evolution of the amplitude of intermediate wave. The amplitudes for the cases of $\nu=0.4, 0.45, 0.5$, and 0.6 are not significantly different. However, as shown in Fig. 2 there are important differences in the phase plots for these values of ν .

FIGURE 1



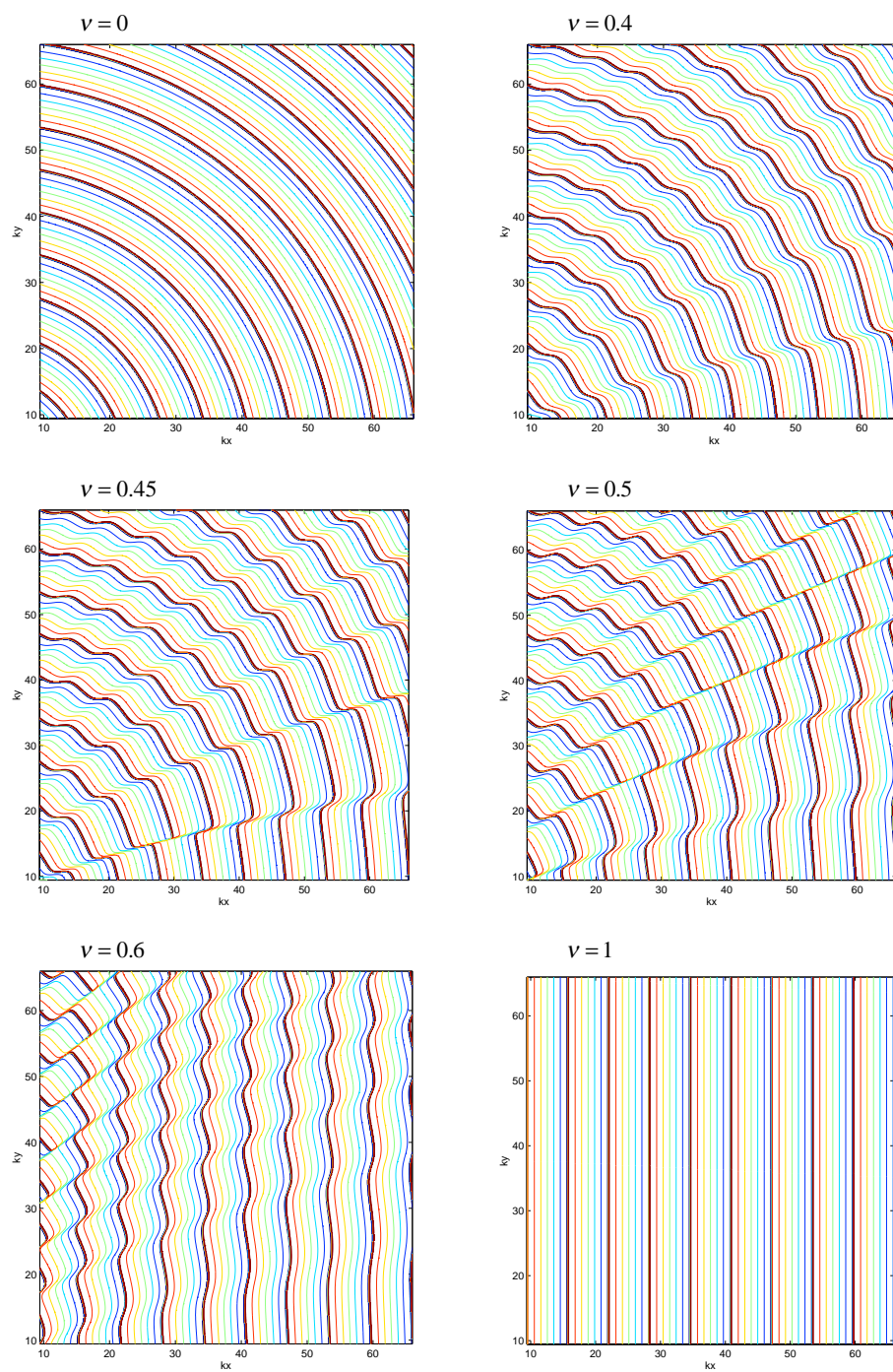


FIGURE 2

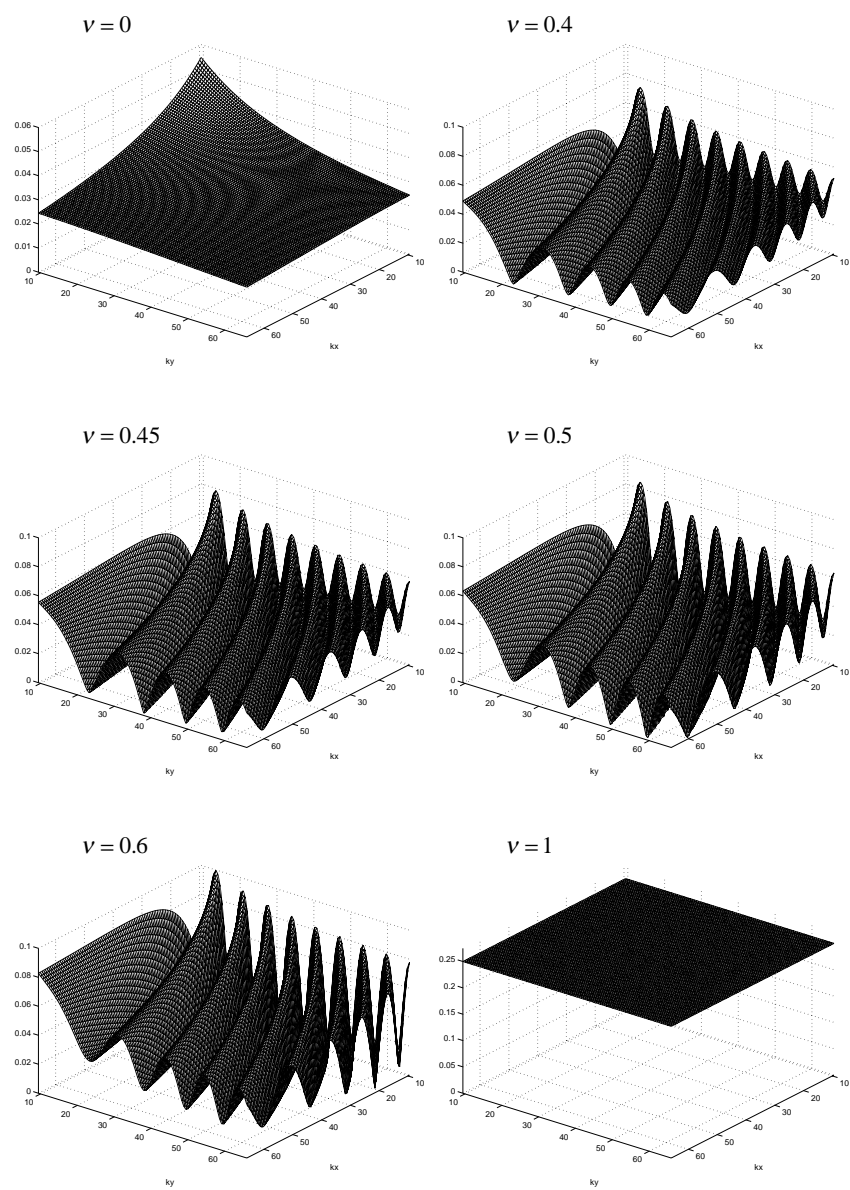


FIGURE 3